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## AN IMPROVED FIRST-APPROXIMATION THEORY FOR THIN SHELLS

By J. LYELL SANDERS, JR.

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**By J. LYELL SANDERS, JR.**

**Langley Research Center  
Langley Field, Va.**

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## AN IMPROVED FIRST-APPROXIMATION THEORY FOR THIN SHELLS

By J. LYELL SANDERS, JR.

### SUMMARY

*An improved form of Love's first-approximation theory for thin shells is derived. In contrast to the results of Love's theory, all strains in the proposed theory vanish for rigid-body motions. Expressions for the stress resultants and couples which satisfy the homogeneous equilibrium equations are given in terms of three stress functions. The special forms of the equations of the new theory in the case of a circular cylinder are given in an appendix.*

### INTRODUCTION

Linear theories of thin shells may be divided roughly into two classes, namely, Love's first approximation and slight modifications of it (refs. 1 and 2), and those theories which stem from Love's second approximation (ref. 1). Most of the papers in the literature which have dealt with the general linear theory of thin shells have been concerned with improvements upon Love's second approximation. An excellent recent survey of the subject is given in reference 3. The present paper is concerned with improving Love's first approximation.

A first-approximation theory for thin shells is distinguished from a second-approximation theory in that in the former theory the effects of transverse shear and normal strain are neglected. The latest forms of Love's first-approximation theory (as set forth in refs. 2 and 4) still contain an inconsistency which the present paper removes. The inconsistency in the equations of the theory is that, except for the special case of axisymmetric loading of shells of revolution, the strains do not all vanish for small rigid-body rotations of the shell.

In the present analysis a modified first-approximation theory is proposed which removes the inconsistency without complicating the system of equations in any essential way. For simplicity and convenience the theory has been developed almost entirely as a two-dimensional one with use of the principle of virtual work as the main tool in the derivations. The derivation does not follow the method of descent from the three-dimensional equations of elasticity nor is it based on a variational principle.

Results are given in the form based on the use of the lines of curvature as coordinate curves in the middle surface. As a step in the development of an integration theory, a set of three strain compatibility equations is given which lead to expressions for the stress resultants and stress couples in terms of three stress functions. Particular results for a circular cylinder are included.

### SYMBOLS

$h$	shell thickness
$n$	coordinate normal to middle surface
$\bar{n}$	unit normal vector to middle surface
$\bar{r}$	radius vector to middle surface
$\bar{t}_1, \bar{t}_2$	unit tangent vectors to middle surface
$D$	bending stiffness of shell, $\frac{Eh^3}{12(1-\nu^2)}$
$E$	Young's modulus
$M_{11}, M_{22}, M_{12}, M_{21}$	stress couples
$M_x, M_\theta$	stress couples for a circular cylinder
$\bar{M}_{12}$	modified stress couple
$\bar{M}_{x\theta}$	modified stress couple for a circular cylinder
$N_{11}, N_{22}, N_{12}, N_{21}$	stress resultants
$N_x, N_\theta$	stress resultants for a circular cylinder

$\bar{N}_{12}$	modified stress resultant
$\bar{N}_{x\theta}$	modified stress resultant for a circular cylinder
$Q_1, Q_2$	transverse stress resultants
$Q_x, Q_\theta$	transverse stress resultants for a circular cylinder
$R_1, R_2$	principal radii of curvature
$U_1, U_2$	displacements tangential to middle surface
$W$	displacement normal to middle surface
$\alpha_1, \alpha_2$	coefficients in metric form of middle surface
$\gamma_1, \gamma_2$	transverse shear strains
$\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}$	strains in middle surface
$\epsilon_x, \epsilon_\theta, \epsilon_{x\theta}$	strains in middle surface for a circular cylinder
$\kappa_{11}, \kappa_{22}, \kappa_{12}, \kappa_{21}$	bending and twisting strains
$\kappa_x, \kappa_\theta$	bending strains for a circular cylinder
$\bar{\kappa}_{12}$	modified twisting strain
$\bar{\kappa}_{x\theta}$	modified twisting strain for a circular cylinder
$\nu$	Poisson's ratio
$\xi_1, \xi_2$	coordinates on middle surface
$\rho_1, \rho_2, \rho_n$	coefficients defined in equation (A1)
$\phi_1, \phi_2, \phi_n$	rotations
$\chi_1, \chi_2, \psi$	stress functions
$\bar{\Omega}$	constant rotation vector
$\bar{\Delta}$	constant displacement vector

### DERIVATION OF SHELL EQUATIONS GEOMETRY

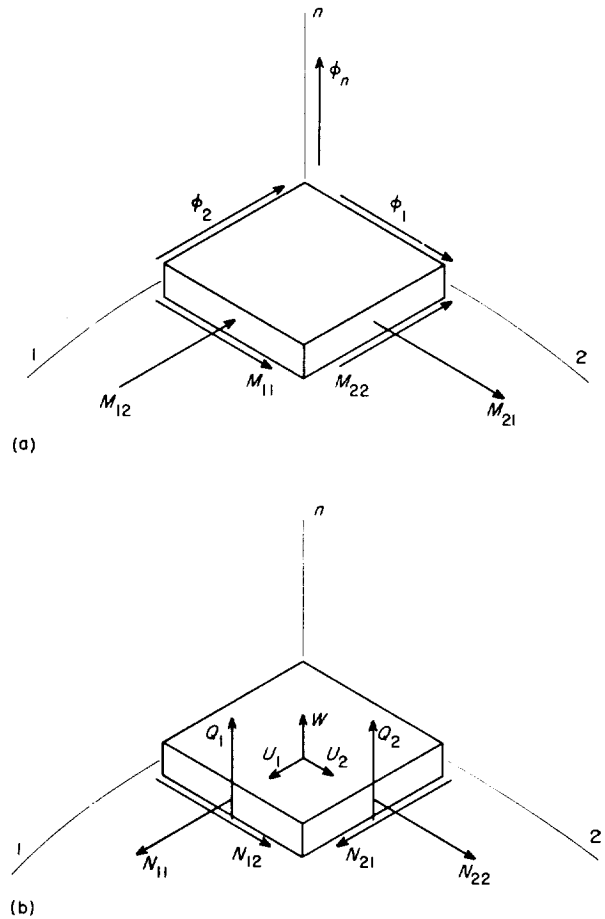
Let the lines of curvature on the middle surface of the shell be used as an orthogonal coordinate net and let  $\xi_1$  and  $\xi_2$  denote the coordinates (as in ref. 2). Let the first fundamental form of the middle surface be given by

$$ds^2 = \alpha_1^2 d\xi_1^2 + \alpha_2^2 d\xi_2^2$$

where  $ds$  is the line element. Let the principal radii of curvature be  $R_1$  and  $R_2$ .

### EQUILIBRIUM EQUATIONS

The 10 stress resultants and couples (or "generalized stresses") which act on sections of the shell parallel to the coordinate curves are shown in figures 1(a) and 1(b). The following six equations of static equilibrium are well-known and



(a) Stress couples and rotations.  
(b) Stress resultants and displacements.

FIGURE 1.—Orientation of coordinates, displacements, rotations, stress resultants, and stress couples.

generally accepted (ref. 2):

$$\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} N_{12} - \frac{\partial \alpha_2}{\partial \xi_1} N_{22} + \frac{\alpha_1 \alpha_2}{R_1} Q_1 = 0 \quad (1)$$

$$\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} N_{21} - \frac{\partial \alpha_1}{\partial \xi_2} N_{11} + \frac{\alpha_1 \alpha_2}{R_2} Q_2 = 0 \quad (2)$$

$$\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \alpha_1 \alpha_2 \left( \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right) = 0 \quad (3)$$

$$\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} M_{12} - \frac{\partial \alpha_2}{\partial \xi_1} M_{22} - \alpha_1 \alpha_2 Q_1 = 0 \quad (4)$$

$$\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} M_{21} - \frac{\partial \alpha_1}{\partial \xi_2} M_{11} - \alpha_1 \alpha_2 Q_2 = 0 \quad (5)$$

$$N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0 \quad (6)$$

Distributed load terms have been omitted for the sake of simplicity. The generalized stresses appearing in these equations may be defined in terms of integrations of three-dimensional stresses through the thickness of the shell as in reference 2. For simplicity in the present development, how-

ever, they are introduced as basic elements of a two-dimensional theory but are otherwise left undefined. Likewise three displacements  $U_1$ ,  $U_2$ , and  $W$  and three rotations  $\phi_1$ ,  $\phi_2$ , and  $\phi_n$  (shown on figs. 1(a) and 1(b)) are introduced as basic quantities of a two-dimensional theory but are otherwise left undefined.

#### STRAIN-DISPLACEMENT RELATIONS

A set of 10 generalized strain quantities (one corresponding to each generalized stress) may be derived in terms of the displacements and rotations by means of a principle of virtual work. In

the present case this principle is

$$\begin{aligned} \iint \left\{ \left( \frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} N_{12} - \frac{\partial \alpha_2}{\partial \xi_1} N_{22} + \frac{\alpha_1 \alpha_2}{R_1} Q_1 \right) \delta U_1 + \left( \frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} N_{21} - \frac{\partial \alpha_1}{\partial \xi_2} N_{11} + \frac{\alpha_1 \alpha_2}{R_2} Q_2 \right) \delta U_2 \right. \\ \left. + \left[ \frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \alpha_1 \alpha_2 \left( \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right) \right] \delta W + \left( \frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} M_{12} - \frac{\partial \alpha_2}{\partial \xi_1} M_{22} - \alpha_1 \alpha_2 Q_1 \right) \delta \phi_1 \right. \\ \left. + \left( \frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} M_{21} - \frac{\partial \alpha_1}{\partial \xi_2} M_{11} - \alpha_1 \alpha_2 Q_2 \right) \delta \phi_2 + \alpha_1 \alpha_2 \left( N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} \right) \delta \phi_n \right\} d\xi_1 d\xi_2 = 0 \quad (7) \end{aligned}$$

This integral vanishes by virtue of equations (1) to (6). Integrating by parts yields

$$\begin{aligned} \oint_C [(N_{11} \delta U_1 + N_{12} \delta U_2 + Q_1 \delta W + M_{11} \delta \phi_1 + M_{12} \delta \phi_2) \alpha_2 d\xi_2 - (N_{21} \delta U_1 + N_{22} \delta U_2 + Q_2 \delta W + M_{21} \delta \phi_1 + M_{22} \delta \phi_2) \alpha_1 d\xi_1] \\ - \iint \left\{ N_{11} \delta \left( \alpha_2 \frac{\partial U_1}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2} U_2 + \frac{\alpha_1 \alpha_2}{R_1} W \right) + N_{12} \delta \left( \alpha_2 \frac{\partial U_2}{\partial \xi_1} - \frac{\partial \alpha_1}{\partial \xi_2} U_1 - \alpha_1 \alpha_2 \phi_n \right) \right. \\ \left. + N_{21} \delta \left( \alpha_1 \frac{\partial U_1}{\partial \xi_2} - \frac{\partial \alpha_2}{\partial \xi_1} U_2 + \alpha_1 \alpha_2 \phi_n \right) + N_{22} \delta \left( \alpha_1 \frac{\partial U_2}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} U_1 + \frac{\alpha_1 \alpha_2}{R_2} W \right) + Q_1 \delta \left( \alpha_2 \frac{\partial W}{\partial \xi_1} - \frac{\alpha_1 \alpha_2}{R_1} U_1 + \alpha_1 \alpha_2 \phi_1 \right) \right. \\ \left. + Q_2 \delta \left( \alpha_1 \frac{\partial W}{\partial \xi_2} - \frac{\alpha_1 \alpha_2}{R_2} U_2 + \alpha_1 \alpha_2 \phi_2 \right) + M_{11} \delta \left( \alpha_2 \frac{\partial \phi_1}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2} \phi_2 \right) + M_{12} \delta \left( \alpha_2 \frac{\partial \phi_2}{\partial \xi_1} - \frac{\partial \alpha_1}{\partial \xi_2} \phi_1 - \frac{\alpha_1 \alpha_2}{R_1} \phi_n \right) \right. \\ \left. + M_{21} \delta \left( \alpha_1 \frac{\partial \phi_1}{\partial \xi_2} - \frac{\partial \alpha_2}{\partial \xi_1} \phi_2 + \frac{\alpha_1 \alpha_2}{R_2} \phi_n \right) + M_{22} \delta \left( \alpha_1 \frac{\partial \phi_2}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} \phi_1 \right) \right\} d\xi_1 d\xi_2 = 0 \quad (8) \end{aligned}$$

where the double integral extends over a region of the middle surface of the shell enclosed by the curve C. Of the two integrals in equation (8) (which equal each other), the first represents the virtual work of the forces acting on the boundary C, and the second must therefore represent the virtual change in strain energy of the portion of the shell within C. These considerations lead to the

following definitions for the 10 strain quantities:

$$\epsilon_{11} = \frac{1}{\alpha_1} \frac{\partial U_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} U_2 + \frac{W}{R_1} \quad (9)$$

$$\epsilon_{22} = \frac{1}{\alpha_2} \frac{\partial U_2}{\partial \xi_2} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} U_1 + \frac{W}{R_2} \quad (10)$$

$$\epsilon_{12} = \frac{1}{\alpha_1} \frac{\partial U_2}{\partial \xi_1} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} U_1 - \phi_n \quad (11)$$

$$\epsilon_{21} = \frac{1}{\alpha_2} \frac{\partial U_1}{\partial \xi_2} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} U_2 + \phi_n \quad (12)$$

$$\kappa_{11} = \frac{1}{\alpha_1} \frac{\partial \phi_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \phi_2 \quad (13)$$

$$\kappa_{22} = \frac{1}{\alpha_2} \frac{\partial \phi_2}{\partial \xi_2} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} \phi_1 \quad (14)$$

$$\kappa_{12} = \frac{1}{\alpha_1} \frac{\partial \phi_2}{\partial \xi_1} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \phi_1 - \frac{\phi_n}{R_1} \quad (15)$$

$$\kappa_{21} = \frac{1}{\alpha_2} \frac{\partial \phi_1}{\partial \xi_2} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} \phi_2 + \frac{\phi_n}{R_2} \quad (16)$$

$$\gamma_1 = \frac{1}{\alpha_1} \frac{\partial W}{\partial \xi_1} - \frac{U_1}{R_1} + \phi_1 \quad (17)$$

$$\gamma_2 = \frac{1}{\alpha_2} \frac{\partial W}{\partial \xi_2} - \frac{U_2}{R_2} + \phi_2 \quad (18)$$

Each of these 10 strain quantities vanishes for small rigid-body motions as will be shown in appendix A.

#### SIMPLIFICATION OF RESULTS

The quantities  $\gamma_1$  and  $\gamma_2$  are transverse-shear strains corresponding to the transverse-shear stress resultants  $Q_1$  and  $Q_2$ . In the theory derived herein, these two strains are neglected. Setting  $\gamma_1 = \gamma_2 = 0$  gives the following expressions for the rotations  $\phi_1$  and  $\phi_2$  in terms of  $U_1$ ,  $U_2$ , and  $W$ :

$$\phi_1 = \frac{U_1}{R_1} - \frac{1}{\alpha_1} \frac{\partial W}{\partial \xi_1} \quad (19)$$

$$\phi_2 = \frac{U_2}{R_2} - \frac{1}{\alpha_2} \frac{\partial W}{\partial \xi_2} \quad (20)$$

The rotation about the normal  $\phi_n$  may be calculated in terms of  $U_1$  and  $U_2$  by several methods (for example, by taking the surface curl of a displacement vector, see ref. 5); in any case, the result is

$$\phi_n = \frac{1}{2\alpha_1\alpha_2} \left( \frac{\partial \alpha_2 U_2}{\partial \xi_1} - \frac{\partial \alpha_1 U_1}{\partial \xi_2} \right) \quad (21)$$

From a comparison of equations (11), (12), and (21) it follows that

$$\epsilon_{12} = \epsilon_{21} \quad (22)$$

The definitions (eqs. (19), (20), and (21)) for  $\phi_1$ ,  $\phi_2$ , and  $\phi_n$  taken together with the Codazzi

relations (see ref. 1, p. 517, eqs. (10))

$$\left. \begin{aligned} \frac{\partial}{\partial \xi_1} \left( \frac{\alpha_2}{R_2} \right) &= \frac{1}{R_1} \frac{\partial \alpha_2}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \left( \frac{\alpha_1}{R_1} \right) &= \frac{1}{R_2} \frac{\partial \alpha_1}{\partial \xi_2} \end{aligned} \right\} \quad (23)$$

and the definitions (eqs. (15) and (16)) for  $\kappa_{12}$  and  $\kappa_{21}$  yield the following identity:

$$\kappa_{12} - \kappa_{21} = \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) (\epsilon_{12} + \epsilon_{21}) = \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \epsilon_{12} \quad (24)$$

The area integral in equation (8) may be written

$$\iint (N_{11} \delta \epsilon_{11} + N_{12} \delta \epsilon_{12} + N_{21} \delta \epsilon_{21} + N_{22} \delta \epsilon_{22} + M_{11} \delta \kappa_{11} + M_{12} \delta \kappa_{12} + M_{21} \delta \kappa_{21} + M_{22} \delta \kappa_{22}) \alpha_1 \alpha_2 d\xi_1 d\xi_2 \quad (25)$$

By using equations (22) and (24) this integral may be written

$$\iint \left\{ N_{11} \delta \epsilon_{11} + 2 \left[ \frac{1}{2} (N_{12} + N_{21}) + \frac{1}{4} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) (M_{12} - M_{21}) \right] \delta \epsilon_{12} + N_{22} \delta \epsilon_{22} + M_{11} \delta \kappa_{11} + \frac{1}{2} (M_{12} + M_{21}) \delta (\kappa_{12} + \kappa_{21}) + M_{22} \delta \kappa_{22} \right\} \alpha_1 \alpha_2 d\xi_1 d\xi_2 \quad (26)$$

Let

$$\bar{N}_{12} = \frac{1}{2} (N_{12} + N_{21}) \quad (27)$$

$$\bar{M}_{12} = \frac{1}{2} (M_{12} + M_{21}) \quad (28)$$

$$\bar{\kappa}_{12} = \frac{1}{2} (\kappa_{12} + \kappa_{21}) \quad (29)$$

If the term  $M_{12} - M_{21}$  is neglected, the expression (26) then becomes

$$\iint (N_{11} \delta \epsilon_{11} + 2 \bar{N}_{12} \delta \epsilon_{12} + N_{22} \delta \epsilon_{22} + M_{11} \delta \kappa_{11} + 2 \bar{M}_{12} \delta \bar{\kappa}_{12} + M_{22} \delta \kappa_{22}) \alpha_1 \alpha_2 d\xi_1 d\xi_2 \quad (30)$$

Note that  $N_{12}$  and  $N_{21}$  occur in expression (26) only in the combination  $\frac{1}{2} (N_{12} + N_{21})$  and that  $N_{12} - N_{21}$  is not being neglected in expression (30).

The quantity  $M_{12}-M_{21}$  need not necessarily be neglected in the present derivation because  $\bar{N}_{12}$  could be defined to be the whole coefficient of  $2\delta\epsilon_{12}$  in expression (26); however, the simplification (27) seems to be justified because  $\frac{1}{4}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)(M_{12}-M_{21})$  would ordinarily be small compared with  $\frac{1}{2}(N_{12}+N_{21})$ .

#### MODIFIED EQUILIBRIUM EQUATIONS

The number of stress quantities aside from  $Q_1$  and  $Q_2$  has been reduced from eight to six (as in Love's first approximation) but the equilibrium equations (1) to (6) are obviously in need of some modification. In the usual derivation of the equations of Love's first approximation theory, the distinction between  $N_{12}$  and  $N_{21}$  and between  $M_{12}$  and  $M_{21}$  is dropped and equation (6) is suppressed.

It is argued that equation (6) is satisfied identically by exact expressions for the stress resultants and couples (in terms of integrals of stress through the thickness of the shell). However, the exact expressions are not actually used in the theory. In the present theory a modified set of equilibrium equations is derived by another application of the principle of virtual work starting from an energy expression equivalent to expression (30).

Since all the strain quantities are expressible in terms of  $U_1$ ,  $U_2$ , and  $W$ , such expressions could be introduced into equation (30), in which case an application of the principle of virtual work would lead to three equilibrium equations. A slightly different procedure leading to essentially equivalent results is used instead. It is convenient to reintroduce the quantities  $Q_1$  and  $Q_2$  and to use  $\phi_1$  and  $\phi_2$  as they are rather than to express them in terms of  $U_1$ ,  $U_2$ , and  $W$ . The expression for the virtual change in strain energy may then

be written as

$$\begin{aligned}
 & \iint \left\{ N_{11}\delta \left( \alpha_2 \frac{\partial U_1}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2} U_2 + \frac{\alpha_1 \alpha_2}{R_1} W \right) + N_{22}\delta \left( \alpha_1 \frac{\partial U_2}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} U_1 + \frac{\alpha_1 \alpha_2}{R_2} W \right) \right. \\
 & + \bar{N}_{12}\delta \left( \alpha_2 \frac{\partial U_2}{\partial \xi_1} + \alpha_1 \frac{\partial U_1}{\partial \xi_2} - \frac{\partial \alpha_1}{\partial \xi_2} U_1 - \frac{\partial \alpha_2}{\partial \xi_1} U_2 \right) + M_{11}\delta \left( \alpha_2 \frac{\partial \phi_1}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2} \phi_2 \right) \\
 & + \bar{M}_{12}\delta \left[ \alpha_2 \frac{\partial \phi_2}{\partial \xi_1} + \alpha_1 \frac{\partial \phi_1}{\partial \xi_2} - \frac{\partial \alpha_1}{\partial \xi_2} \phi_1 - \frac{\partial \alpha_2}{\partial \xi_1} \phi_2 + \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \left( \alpha_2 \frac{\partial U_2}{\partial \xi_1} - \alpha_1 \frac{\partial U_1}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} U_2 - \frac{\partial \alpha_1}{\partial \xi_2} U_1 \right) \right] \\
 & + M_{22}\delta \left( \alpha_1 \frac{\partial \phi_2}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} \phi_1 \right) + Q_1\delta \left( \alpha_2 \frac{\partial W}{\partial \xi_1} - \frac{\alpha_1 \alpha_2}{R_1} U_1 + \alpha_1 \alpha_2 \phi_1 \right) + Q_2\delta \left( \alpha_1 \frac{\partial W}{\partial \xi_2} - \frac{\alpha_1 \alpha_2}{R_2} U_2 + \alpha_1 \alpha_2 \phi_2 \right) \Big\} d\xi_1 d\xi_2 \\
 & = \oint_C \left[ \left\{ N_{11}\delta U_1 + \bar{N}_{12}\delta U_2 + Q_1\delta W + M_{11}\delta \phi_1 + \bar{M}_{12}\delta \left[ \phi_2 + \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) U_2 \right] \right\} \alpha_2 d\xi_2 \right. \\
 & - \left\{ N_{22}\delta U_2 + \bar{N}_{12}\delta U_1 + Q_2\delta W + M_{22}\delta \phi_2 + \bar{M}_{12}\delta \left[ \phi_1 - \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) U_1 \right] \right\} \alpha_1 d\xi_1 \Big] \\
 & - \iint \left\{ \left[ \frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 \bar{N}_{12}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} \bar{N}_{12} - \frac{\partial \alpha_2}{\partial \xi_1} N_{22} + \frac{\alpha_1 \alpha_2}{R_1} Q_1 + \frac{\alpha_1}{2} \frac{\partial}{\partial \xi_2} \left( \frac{\bar{M}_{12}}{R_1} \right) - \frac{\alpha_1}{2} \frac{\partial}{\partial \xi_2} \left( \frac{\bar{M}_{12}}{R_2} \right) \right] \delta U_1 \right. \\
 & \quad \left. + [\dots] \delta U_2 + [\dots] \delta W + [\dots] \delta \phi_1 + [\dots] \delta \phi_2 \right\} d\xi_1 d\xi_2 \quad (31)
 \end{aligned}$$

If the portion of the shell within C is in equilibrium, by the principle of virtual work the left-hand side of expression (31) must equal the line integral on the right-hand side, in which case the area integral on the right must vanish. Since the virtual displacements may be assumed to be arbitrary

and independent, the following conditions of equilibrium must hold:

$$\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 \bar{N}_{12}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} \bar{N}_{12} - \frac{\partial \alpha_2}{\partial \xi_1} N_{22} + \frac{\alpha_1 \alpha_2}{R_1} Q_1 + \frac{\alpha_1}{2} \frac{\partial}{\partial \xi_2} \left[ \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \bar{M}_{12} \right] = 0 \quad (32)$$

$$\frac{\partial \alpha_2 \bar{N}_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} \bar{N}_{12} - \frac{\partial \alpha_1}{\partial \xi_2} N_{11} + \frac{\alpha_1 \alpha_2}{R_2} Q_2 + \frac{\alpha_2}{2} \frac{\partial}{\partial \xi_1} \left[ \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \bar{M}_{12} \right] = 0 \quad (33)$$

$$\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \left( \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right) \alpha_1 \alpha_2 = 0 \quad (34)$$

$$\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 \bar{M}_{12}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} \bar{M}_{12} - \frac{\partial \alpha_2}{\partial \xi_1} M_{22} - \alpha_1 \alpha_2 Q_1 = 0 \quad (35)$$

$$\frac{\partial \alpha_2 \bar{M}_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} \bar{M}_{12} - \frac{\partial \alpha_1}{\partial \xi_2} M_{11} - \alpha_1 \alpha_2 Q_2 = 0 \quad (36)$$

These equations (eqs. (32) to (36)) are the equilibrium equations of the proposed new theory.

#### STRESS-STRAIN RELATIONS

The set of stress-strain relations which is appropriate depends on the mechanical properties of the material of the shell which are not necessarily elastic. In the important case of an isotropic elastic material, the stress-strain relations may be taken to be the same as those in Love's first approximation, namely:

$$\left. \begin{aligned} Eh\epsilon_{11} &= N_{11} - \nu N_{22} & \frac{Eh^3}{12} \kappa_{11} &= M_{11} - \nu M_{22} \\ Eh\epsilon_{22} &= N_{22} - \nu N_{11} & \frac{Eh^3}{12} \kappa_{22} &= M_{22} - \nu M_{11} \\ Eh\epsilon_{12} &= (1 + \nu) \bar{N}_{12} & \frac{Eh^3}{12} \bar{\kappa}_{12} &= (1 + \nu) \bar{M}_{12} \end{aligned} \right\} \quad (37)$$

where  $h$  is the shell thickness,  $E$  is Young's modulus, and  $\nu$  is Poisson's ratio.

#### MODIFIED STRAIN-DISPLACEMENT RELATIONS

For convenience, the strain-displacement rela-

tions of the new theory are repeated here.

$$\left. \begin{aligned} \epsilon_{11} &= \frac{1}{\alpha_1} \frac{\partial U_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} U_2 + \frac{W}{R_1} \\ \epsilon_{22} &= \frac{1}{\alpha_2} \frac{\partial U_2}{\partial \xi_2} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} U_1 + \frac{W}{R_2} \\ \epsilon_{12} &= \frac{1}{2\alpha_1 \alpha_2} \left( \alpha_2 \frac{\partial U_2}{\partial \xi_1} + \alpha_1 \frac{\partial U_1}{\partial \xi_2} - \frac{\partial \alpha_1}{\partial \xi_2} U_1 - \frac{\partial \alpha_2}{\partial \xi_1} U_2 \right) \\ \kappa_{11} &= \frac{1}{\alpha_1} \frac{\partial \phi_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \phi_2 \\ \kappa_{22} &= \frac{1}{\alpha_2} \frac{\partial \phi_2}{\partial \xi_2} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} \phi_1 \\ \bar{\kappa}_{12} &= \frac{1}{2\alpha_1 \alpha_2} \left[ \alpha_2 \frac{\partial \phi_2}{\partial \xi_1} + \alpha_1 \frac{\partial \phi_1}{\partial \xi_2} - \frac{\partial \alpha_1}{\partial \xi_2} \phi_1 - \frac{\partial \alpha_2}{\partial \xi_1} \phi_2 \right. \\ &\quad \left. + \frac{1}{2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \left( \frac{\partial \alpha_2 U_2}{\partial \xi_1} - \frac{\partial \alpha_1 U_1}{\partial \xi_2} \right) \right] \end{aligned} \right\} \quad (38)$$

It is shown in appendix A that each of these strain quantities vanishes for small rigid-body motions.

#### BOUNDARY CONDITIONS

The boundary integral on the right-hand side of equation (31) may be used to determine the proper form of boundary conditions to be applied in the present theory. Since the process of deriving them is well-known, only the results will be given here.

On an edge where  $\xi_1$  is a constant prescribe:

$$N_{11} + \frac{1}{R_1} M_{11} \quad \text{or} \quad U_1 \quad (39a)$$

$$\bar{N}_{12} + \left( \frac{3}{2R_2} - \frac{1}{2R_1} \right) \bar{M}_{12} \quad \text{or} \quad U_2 \quad (39b)$$

$$Q_1 + \frac{1}{\alpha_2} \frac{\partial \bar{M}_{12}}{\partial \xi_2} \quad \text{or} \quad W \quad (39c)$$

$$M_{11} \quad \text{or} \quad \frac{\partial W}{\partial \xi_1} \quad (39d)$$

On an edge where  $\xi_2$  is a constant, the conditions are the same as those in expressions (39) except that the subscripts 1 and 2 are interchanged.

#### COMPATIBILITY EQUATIONS AND STRESS FUNCTIONS

The six strain quantities  $\epsilon_{11}$ ,  $\epsilon_{22}$ ,  $\epsilon_{12}$ ,  $\kappa_{11}$ ,  $\kappa_{22}$ , and

$\bar{\kappa}_{12}$  (eqs. (38)) satisfy identically the following compatibility equations:

$$\frac{\partial \alpha_2 \kappa_{22}}{\partial \xi_1} - \frac{\partial \alpha_1 \bar{\kappa}_1}{\partial \xi_2} - \frac{\partial \alpha_1}{\partial \xi_2} \bar{\kappa}_{12} - \frac{\partial \alpha_2}{\partial \xi_1} \kappa_{11} + \frac{1}{R_1} \left( -\frac{\partial \alpha_1 \epsilon_{12}}{\partial \xi_1} + \frac{\partial \alpha_2 \epsilon_{11}}{\partial \xi_2} + \frac{\partial \alpha_1}{\partial \xi_2} \epsilon_{11} + \frac{\partial \alpha_2}{\partial \xi_1} \epsilon_{12} \right) + \frac{\alpha_1}{2} \frac{\partial}{\partial \xi_2} \left[ \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \epsilon_{12} \right] = 0 \quad (40)$$

$$\frac{\partial \alpha_1 \kappa_{11}}{\partial \xi_2} - \frac{\partial \alpha_2 \bar{\kappa}_{12}}{\partial \xi_1} - \frac{\partial \alpha_2}{\partial \xi_1} \bar{\kappa}_{12} - \frac{\partial \alpha_1}{\partial \xi_2} \kappa_{22} + \frac{1}{R_2} \left( -\frac{\partial \alpha_1 \epsilon_{11}}{\partial \xi_2} + \frac{\partial \alpha_2 \epsilon_{12}}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2} \epsilon_{22} + \frac{\partial \alpha_2}{\partial \xi_1} \epsilon_{12} \right) + \frac{\alpha_2}{2} \frac{\partial}{\partial \xi_1} \left[ \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \epsilon_{12} \right] = 0 \quad (41)$$

$$\begin{aligned} & \frac{\partial}{\partial \xi_1} \left[ \frac{1}{\alpha_1} \left( -\frac{\partial \alpha_2 \epsilon_{22}}{\partial \xi_1} + \frac{\partial \alpha_1 \epsilon_{12}}{\partial \xi_2} + \frac{\partial \alpha_2}{\partial \xi_1} \epsilon_{11} + \frac{\partial \alpha_1}{\partial \xi_2} \epsilon_{12} \right) \right] + \frac{\partial}{\partial \xi_2} \\ & \left[ \frac{1}{\alpha_2} \left( -\frac{\partial \alpha_1 \epsilon_{11}}{\partial \xi_2} + \frac{\partial \alpha_2 \epsilon_{12}}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2} \epsilon_{22} + \frac{\partial \alpha_2}{\partial \xi_1} \epsilon_{12} \right) \right] - \alpha_1 \alpha_2 \left( \frac{\kappa_{11}}{R_2} + \frac{\kappa_{22}}{R_1} \right) = 0 \quad (42) \end{aligned}$$

If in equations (40) to (42)  $\epsilon_{11}$  is replaced by  $-M_{22}$ ,  $\epsilon_{22}$  is replaced by  $-M_{11}$ ,  $\epsilon_{12}$  is replaced by  $\bar{M}_{12}$ ,  $\kappa_{11}$  is replaced by  $N_{22}$ ,  $\kappa_{22}$  is replaced by  $N_{11}$ , and  $\bar{\kappa}_{12}$  is replaced by  $-\bar{N}_{12}$ , then these equations become identical to the modified equilibrium equations (eqs. (32) to (36)) written in the form with  $Q_1$  and  $Q_2$  eliminated. Because of this remarkable circumstance expressions for the stress resultants and couples which satisfy the homogeneous equilibrium equations identically can be written down by inspection by using the strain-displacement relations (38) and making these replacements. Three stress functions corresponding to the three displacements are introduced. Let  $x_1$  correspond to  $-U_1$ ,  $x_2$  correspond to  $-U_2$ , and  $\psi$  correspond to  $-W$ . The following expressions then satisfy the equilibrium equations identically:

$$N_{11} = \frac{1}{\alpha_2} \frac{\partial}{\partial \xi_2} \left( \frac{1}{\alpha_2} \frac{\partial \psi}{\partial \xi_2} - \frac{x_2}{R_2} \right) + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} \left( \frac{1}{\alpha_1} \frac{\partial \psi}{\partial \xi_1} - \frac{x_1}{R_1} \right) \quad (43)$$

$$N_{22} = \frac{1}{\alpha_1} \frac{\partial}{\partial \xi_1} \left( \frac{1}{\alpha_1} \frac{\partial \psi}{\partial \xi_1} - \frac{x_1}{R_1} \right) + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \left( \frac{1}{\alpha_2} \frac{\partial \psi}{\partial \xi_2} - \frac{x_2}{R_2} \right) \quad (44)$$

$$\begin{aligned} \bar{N}_{12} = & \frac{1}{2\alpha_1} \frac{\partial}{\partial \xi_1} \left( \frac{x_2}{R_2} - \frac{1}{\alpha_2} \frac{\partial \psi}{\partial \xi_2} \right) + \frac{1}{2\alpha_2} \frac{\partial}{\partial \xi_2} \left( \frac{x_1}{R_1} - \frac{1}{\alpha_1} \frac{\partial \psi}{\partial \xi_1} \right) \\ & - \frac{1}{2\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \left( \frac{x_1}{R_1} - \frac{1}{\alpha_1} \frac{\partial \psi}{\partial \xi_1} \right) - \frac{1}{2\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} \left( \frac{x_2}{R_2} - \frac{1}{\alpha_2} \frac{\partial \psi}{\partial \xi_2} \right) \\ & + \frac{1}{4\alpha_1 \alpha_2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \left( \frac{\partial \alpha_2 x_2}{\partial \xi_1} - \frac{\partial \alpha_1 x_1}{\partial \xi_2} \right) \quad (45) \end{aligned}$$

$$M_{11} = \frac{1}{\alpha_2} \frac{\partial x_2}{\partial \xi_2} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi_1} x_1 + \frac{\psi}{R_2} \quad (46)$$

$$M_{22} = \frac{1}{\alpha_1} \frac{\partial x_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} x_2 + \frac{\psi}{R_1} \quad (47)$$

$$\bar{M}_{12} = -\frac{1}{2\alpha_1} \frac{\partial x_2}{\partial \xi_1} - \frac{1}{2\alpha_2} \frac{\partial x_1}{\partial \xi_2} + \frac{1}{2\alpha_1 \alpha_2} \left( \frac{\partial \alpha_1}{\partial \xi_2} x_1 + \frac{\partial \alpha_2}{\partial \xi_1} x_2 \right) \quad (48)$$

The essence of the new theory is contained in equations (32) to (48). The particular forms of these equations which are appropriate to circular cylindrical shells are presented in appendix B.

#### CONCLUDING REMARKS

An improved first-approximation theory for thin shells has been derived. The strain-displacement relations are more realistic than those of Love's first approximation because all strains vanish for small rigid-body motions of the shell whereas for Love's theory they do not.

In previous derivations of a first-approximation theory the number of unknown stress resultants and couples are reduced from 10 to 8 by making approximations in the expressions for the resultants in terms of integrals of stress through the thickness of the shell. In the present derivation the reduction in number of stress unknowns from 10 to 8 is made by combining some of them in a way suggested by a certain expression of the principle of virtual work which includes the work done during a small rotation about the normal to the shell. It is not necessary to drop any terms in the expressions for the stress resultants and couples in terms of integrals of stress through the thickness of the shell.

The compatibility equations for the strain quantities involved in the theory lead directly to expressions for the stress resultants and couples in terms of a set of stress functions. These expressions for the stress quantities satisfy the equations of equilibrium identically.

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## APPENDIX A

### SMALL RIGID-BODY MOTIONS

Let  $\bar{r}(\xi_1, \xi_2)$  be the vector from the origin to a point on the middle surface of the shell. If  $\bar{t}_1$  and  $\bar{t}_2$  are unit tangent vectors in the  $\xi_1$  and  $\xi_2$  directions, respectively, and  $\bar{n} = \bar{t}_1 \times \bar{t}_2$  is the unit normal vector,  $\bar{r}$  can be expressed in the form

$$\bar{r} = \rho_1 \bar{t}_1 + \rho_2 \bar{t}_2 + \rho_n \bar{n} \quad (\text{A1})$$

Equation (A1) defines the quantities  $\rho_1$ ,  $\rho_2$ , and  $\rho_n$  as functions of  $\xi_1$  and  $\xi_2$ . The components  $\delta_1$ ,  $\delta_2$ , and  $\delta_n$  of a constant displacement vector  $\bar{\Delta}$  and the components  $\omega_1$ ,  $\omega_2$ , and  $\omega_n$  of a constant (small) rotation vector  $\bar{\Omega}$  are defined in the  $\xi_1, \xi_2, n$  coordinate system by the equations

$$\bar{\Delta} = \delta_1 \bar{t}_1 + \delta_2 \bar{t}_2 + \delta_n \bar{n} \quad (\text{A2})$$

$$\bar{\Omega} = -\omega_2 \bar{t}_1 + \omega_1 \bar{t}_2 + \omega_n \bar{n} \quad (\text{A3})$$

The displacement vector  $\bar{U} = U_1 \bar{t}_1 + U_2 \bar{t}_2 + W \bar{n}$  of points on the middle surface of the shell due to the rigid-body motions  $\bar{\Delta}$  and  $\bar{\Omega}$  is given by

$$\bar{U} = \bar{\Delta} + \bar{\Omega} \times \bar{r} \quad (\text{A4})$$

or in component form by

$$\left. \begin{aligned} U_1 &= \delta_1 + \rho_n \omega_1 - \rho_2 \omega_n \\ U_2 &= \delta_2 + \rho_n \omega_2 + \rho_1 \omega_n \\ W &= \delta_n - \rho_1 \omega_1 - \rho_2 \omega_2 \end{aligned} \right\} \quad (\text{A5})$$

The rotations are given by

$$\left. \begin{aligned} \phi_1 &= \omega_1 \\ \phi_2 &= \omega_2 \\ \phi_n &= \omega_n \end{aligned} \right\} \quad (\text{A6})$$

By using the equations

$$\left. \begin{aligned} \frac{\partial \bar{\Delta}}{\partial \xi_1} = \frac{\partial \bar{\Delta}}{\partial \xi_2} = 0 \quad \frac{\partial \bar{\Omega}}{\partial \xi_1} = \frac{\partial \bar{\Omega}}{\partial \xi_2} = 0 \\ \frac{\partial \bar{r}}{\partial \xi_1} = \alpha_1 \bar{t}_1 \quad \frac{\partial \bar{r}}{\partial \xi_2} = \alpha_2 \bar{t}_2 \end{aligned} \right\} \quad (\text{A7})$$

together with the well-known equations for the derivatives of  $\bar{t}_1$ ,  $\bar{t}_2$ , and  $\bar{n}$  with respect to  $\xi_1$  and  $\xi_2$  (see ref. 2)

$$\left. \begin{aligned} \frac{\partial \bar{t}_1}{\partial \xi_1} &= -\frac{1}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \bar{t}_2 - \frac{\alpha_1}{R_1} \bar{n} & \frac{\partial \bar{t}_1}{\partial \xi_2} &= \frac{1}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} \bar{t}_2 \\ \frac{\partial \bar{t}_2}{\partial \xi_1} &= \frac{1}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \bar{t}_1 & \frac{\partial \bar{t}_2}{\partial \xi_2} &= -\frac{1}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} \bar{t}_1 - \frac{\alpha_2}{R_2} \bar{n} \\ \frac{\partial \bar{n}}{\partial \xi_1} &= \frac{\alpha_1}{R_1} \bar{t}_1 & \frac{\partial \bar{n}}{\partial \xi_2} &= \frac{\alpha_2}{R_2} \bar{t}_2 \end{aligned} \right\}$$

the following formulas may be derived:

$$\left. \begin{aligned} \frac{\partial \rho_1}{\partial \xi_1} &= \frac{\rho_2}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} - \frac{\alpha_1 \rho_n}{R_1} & \frac{\partial \rho_1}{\partial \xi_2} &= \frac{\rho_2}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} \\ \frac{\partial \rho_2}{\partial \xi_1} &= \frac{\rho_1}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} & \frac{\partial \rho_2}{\partial \xi_2} &= \alpha_2 - \frac{\rho_1}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} - \frac{\alpha_2 \rho_n}{R_2} \\ \frac{\partial \rho_n}{\partial \xi_1} &= \frac{\alpha_1 \rho_1}{R_1} & \frac{\partial \rho_n}{\partial \xi_2} &= \frac{\alpha_2 \rho_2}{R_2} \end{aligned} \right\} \quad (\text{A8})$$

$$\left. \begin{aligned} \frac{\partial \delta_1}{\partial \xi_1} &= -\frac{\delta_2}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} - \frac{\alpha_1 \delta_n}{R_1} & \frac{\partial \delta_1}{\partial \xi_2} &= \frac{\delta_2}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} \\ \frac{\partial \delta_2}{\partial \xi_1} &= \frac{\delta_1}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} & \frac{\partial \delta_2}{\partial \xi_2} &= -\frac{\delta_1}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} - \frac{\alpha_2 \delta_n}{R_2} \\ \frac{\partial \delta_n}{\partial \xi_1} &= \frac{\alpha_1 \delta_1}{R_1} & \frac{\partial \delta_n}{\partial \xi_2} &= \frac{\alpha_2 \delta_2}{R_2} \end{aligned} \right\} \quad (\text{A9})$$

$$\left. \begin{aligned} \frac{\partial \omega_1}{\partial \xi_1} &= -\frac{\omega_2}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} & \frac{\partial \omega_1}{\partial \xi_2} &= \frac{\omega_2}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} - \frac{\alpha_2 \omega_n}{R_2} \\ \frac{\partial \omega_2}{\partial \xi_1} &= \frac{\omega_1}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} + \frac{\alpha_1 \omega_n}{R_1} & \frac{\partial \omega_2}{\partial \xi_2} &= -\frac{\omega_1}{\alpha_1} \frac{\partial \alpha_2}{\partial \xi_1} \\ \frac{\partial \omega_n}{\partial \xi_1} &= -\frac{\alpha_1 \omega_2}{R_1} & \frac{\partial \omega_n}{\partial \xi_2} &= \frac{\alpha_2 \omega_1}{R_2} \end{aligned} \right\} \quad (\text{A10})$$

With the use of equations (A8) to (A10) it can be shown that the strains given by equations (9) to (18) or by equations (38) vanish when the rigid-body displacements and rotations given by

equations (A5) and (A6) are imposed. For example, consider  $\kappa_{12}$  (given by eq. (15))

$$\kappa_{12} = \frac{1}{\alpha_1} \frac{\partial \phi_2}{\partial \xi_1} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \phi_1 - \frac{1}{R_1} \phi_n \quad (\text{A11})$$

$$\kappa_{12} = \frac{1}{\alpha_1} \left( \frac{1}{\alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \omega_1 + \frac{\alpha_1}{R_1} \omega_n \right) - \frac{1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi_2} \omega_1 - \frac{1}{R_1} \omega_n \equiv 0 \quad (\text{A12})$$

The same is true for  $\kappa_{21}$  and hence for  $\bar{\kappa}_{12} = \frac{1}{2}(\kappa_{12} + \kappa_{21})$ . The corresponding strain  $\tau$  in Love's theory given by equation (44) of reference 2 as

$$\tau = \frac{1}{\alpha_2} \frac{\partial \phi_1}{\partial \xi_2} + \frac{1}{\alpha_1} \frac{\partial \phi_2}{\partial \xi_1} - \frac{1}{\alpha_1 \alpha_2} \left( \frac{\partial \alpha_1}{\partial \xi_2} \phi_1 + \frac{\partial \alpha_2}{\partial \xi_1} \phi_2 \right) \quad (\text{A13})$$

does not vanish except in the special case of axisymmetric loading of shells of revolution.

## APPENDIX B

### EQUATIONS OF THE THEORY FOR A CIRCULAR CYLINDER

In the case of a circular cylinder let

$$\left. \begin{array}{llll} \xi_1 = x & U_1 = U & R_1 = \infty & \alpha_1 = 1 \\ \xi_2 = \theta & U_2 = V & R_2 = R & \alpha_2 = R \end{array} \right\} \quad (\text{B1})$$

The equilibrium equations are

$$\left. \begin{array}{l} \frac{\partial N_x}{\partial x} + \frac{1}{R} \frac{\partial \bar{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2} \frac{\partial \bar{M}_{x\theta}}{\partial \theta} = 0 \\ \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{1}{R} \frac{\partial N_\theta}{\partial \theta} + \frac{1}{2R} \frac{\partial \bar{M}_{x\theta}}{\partial x} + \frac{1}{h} Q_\theta = 0 \\ \frac{\partial Q_x}{\partial x} + \frac{1}{R} \frac{\partial Q_\theta}{\partial \theta} - \frac{1}{R} N_\theta = 0 \\ \frac{\partial M_x}{\partial x} + \frac{1}{R} \frac{\partial \bar{M}_{x\theta}}{\partial \theta} - Q_x = 0 \\ \frac{\partial \bar{M}_{x\theta}}{\partial x} + \frac{1}{R} \frac{\partial M_\theta}{\partial \theta} - Q_\theta = 0 \end{array} \right\} \quad (\text{B2})$$

The strain-displacement equations are

$$\left. \begin{array}{ll} \epsilon_x = \frac{\partial U}{\partial x} & \kappa_x = -\frac{\partial^2 W}{\partial x^2} \\ \epsilon_\theta = \frac{1}{R} \frac{\partial V}{\partial \theta} + \frac{W}{R} & \kappa_\theta = -\frac{1}{R^2} \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{R^2} \frac{\partial V}{\partial \theta} \\ \epsilon_{x\theta} = \frac{1}{2} \left( \frac{\partial V}{\partial x} + \frac{1}{R} \frac{\partial U}{\partial \theta} \right) & \bar{\kappa}_{x\theta} = -\frac{1}{R} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{3}{4R} \frac{\partial V}{\partial x} - \frac{1}{4R^2} \frac{\partial U}{\partial \theta} \end{array} \right\} \quad (\text{B3})$$

The stress-strain relations are

$$\left. \begin{array}{ll} N_x = \frac{Eh}{1-\nu^2} (\epsilon_x + \nu \epsilon_\theta) & M_x = L' (\kappa_x + \nu \kappa_\theta) \\ N_\theta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu \epsilon_x) & M_\theta = L (\kappa_\theta + \nu \kappa_x) \\ \bar{N}_{x\theta} = \frac{Eh}{1-\nu^2} (1-\nu) \epsilon_{x\theta} & \bar{M}_{x\theta} = L (1-\nu) \bar{\kappa}_{x\theta} \end{array} \right\} \quad (\text{B4})$$

where  $D = \frac{Eh^3}{12(1-\nu^2)}$  and  $h$  is the thickness of the shell. The compatibility equations are

$$\left. \begin{aligned} \frac{\partial \kappa_\theta}{\partial x} - \frac{1}{R} \frac{\partial \bar{\kappa}_{x\theta}}{\partial \theta} - \frac{1}{2R^2} \frac{\partial \epsilon_{x\theta}}{\partial \theta} &= 0 \\ -\frac{\partial \bar{\kappa}_{x\theta}}{\partial x} + \frac{1}{R} \frac{\partial \kappa_x}{\partial \theta} + \frac{3}{2R} \frac{\partial \epsilon_{x\theta}}{\partial x} - \frac{1}{R^2} \frac{\partial \epsilon_x}{\partial \theta} &= 0 \\ -\frac{\partial^2 \epsilon_\theta}{\partial x^2} + \frac{2}{R} \frac{\partial^2 \epsilon_{x\theta}}{\partial x \partial \theta} - \frac{1}{R^2} \frac{\partial^2 \epsilon_x}{\partial \theta^2} - \frac{1}{R} \kappa_x &= 0 \end{aligned} \right\} \quad (B5)$$

The expressions for the stress resultants in terms of stress functions are

$$\left. \begin{aligned} N_x &= \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{R^2} \frac{\partial \chi_2}{\partial \theta} & M_x &= \frac{1}{R} \frac{\partial \chi_2}{\partial \theta} + \frac{\psi}{R} \\ N_\theta &= \frac{\partial^2 \psi}{\partial x^2} & M_\theta &= \frac{\partial \chi_1}{\partial x} \\ \bar{N}_{x\theta} &= -\frac{1}{R} \frac{\partial^2 \psi}{\partial x \partial \theta} + \frac{3}{4R} \frac{\partial \chi_2}{\partial x} - \frac{1}{4R^2} \frac{\partial \chi_1}{\partial \theta} & \bar{M}_{x\theta} &= -\frac{1}{2} \left( \frac{1}{R} \frac{\partial \chi_1}{\partial \theta} + \frac{\partial \chi_2}{\partial x} \right) \end{aligned} \right\} \quad (B6)$$

Some comparison with other theories of circular cylinders is afforded by the following means. Let

$$\left. \begin{aligned} U &= \bar{U} e^{\frac{\lambda x}{R}} \cos n\theta \\ V &= \bar{V} e^{\frac{\lambda x}{R}} \sin n\theta \\ W &= \bar{W} e^{\frac{\lambda x}{R}} \cos n\theta \end{aligned} \right\} \quad (B7)$$

where  $n$  is the number of the harmonic. Use of equations (B7), (B3), (B4), and (B2) leads to the following equation for the determination of  $\lambda$ :

$$\lambda^8 - 4n^2\lambda^6 + \left[ \frac{1-\nu^2}{K} + 6n^2(n^2-1) \right] \lambda^4 - 4n^2(n^2-1)^2\lambda^2 + n^4(n^2-1)^2 = 0 \quad (B8)$$

where  $K = \frac{h^2}{12R^2}$ . Terms of higher order in  $K$  than

those shown have been neglected in equation (B8). The corresponding equations for  $\lambda$  for several other theories of circular cylindrical shells are given by Naghdi and Berry in reference 6. For instance, from Love's theory the following equation is obtained:

$$\lambda^8 - 4n^2\lambda^6 + \left[ \frac{1-\nu^2}{K} + 6n^4 - 2(2+\nu)n^2 \right] \lambda^4 + \left[ -4n^6 + 2(3+\nu)n^4 - \frac{5+3\nu}{2}n^2 \right] \lambda^2 + n^4(n^2-1)^2 = 0$$

and Flügge's theory yields

$$\lambda^8 - 2(2n^2-\nu)\lambda^6 + \left( \frac{1-\nu^2}{K} + 6n^4 - 6n^2 \right) \lambda^4 + [-4n^6 + 2(4-\nu)n^4 - 2(2-\nu)n^2] \lambda^2 + n^4(n^2-1)^2 = 0$$

Equation (B8) is remarkably free of Poisson's ratio terms compared with these equations and the corresponding equations for other theories.

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